TRICorporation

Experiential Leadership and Simulation Programs

Improved decision-making. Enhanced performance. Exceptional results.

Time Value of Money

Critical Equation #10 for Business Leaders

$(1+R)^N$ *et al.*

Overview

The time value of money is fundamental to all aspects of business decision-making. Consider the following: ***"Would you rather have $100 today or $200 10 years from now?"*** As simple as this question might seem, there are complex factors that can influence your choice. Our purpose is to explore and simplify the concept of time value of money, providing the insight and tools you need to answer this type of question and, ultimately, create shareholder value.

One aspect of time value of money that is critical to long-term growth is compounding. Einstein has been quoted as saying ***"The most powerful force in the universe is compound interest."*** Others have referred to compounding as the ***"8th wonder of the world."*** While both quotes are anecdotal at best, they do emphasize the importance of understanding the time value of money.

In the sections that follow, we will:

- present in detail TRI's Critical Equation # 10 on time value of money,
- define both future and present value,
- provide a simple formulation for solving time value of money problems,
- examine a variety of applications for decision-makers, and
- bring it back to work throughout.

Finally, we will provide references to our other TRI Critical Equations as appropriate.

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Preparing to Solve Time Value of Money Problems

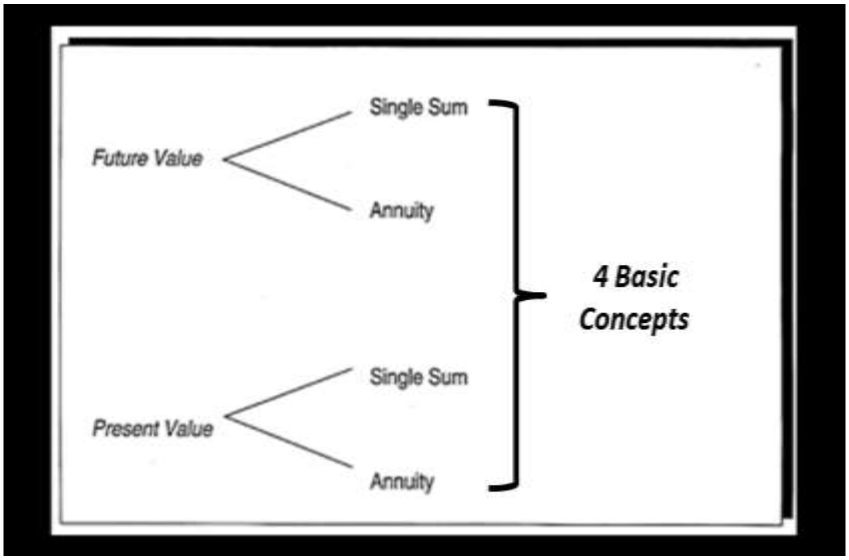
In this section, we will introduce the basic concept of time value of money and the structure for solving a problem. There are two basic classifications of value in this domain: present value and future value. Future value refers to a scenario in which an investment is made today and we need to determine what the value will be at a point in the future. Present value comes into play when you have the right to receive money in the future and would like to understand its value today.

Within each classification there are two sub-classifications. The first is a single sum, which is simply $X either today or at a future point in time (e.g., $100 to be received in 25 years; $200 to be invested today). The second is an annuity. An annuity in its most basic form is a constant dollar amount per a stated time period (e.g., $300 invested per year for five years; $400 received per year for the next 10 years). Key phrases that let you know you are dealing with an annuity are ***"per year"*** or ***"per stated time period."***

Exhibit 1 summarizes the four types of time value of money scenarios that business leaders need to understand.

Exhibit 1

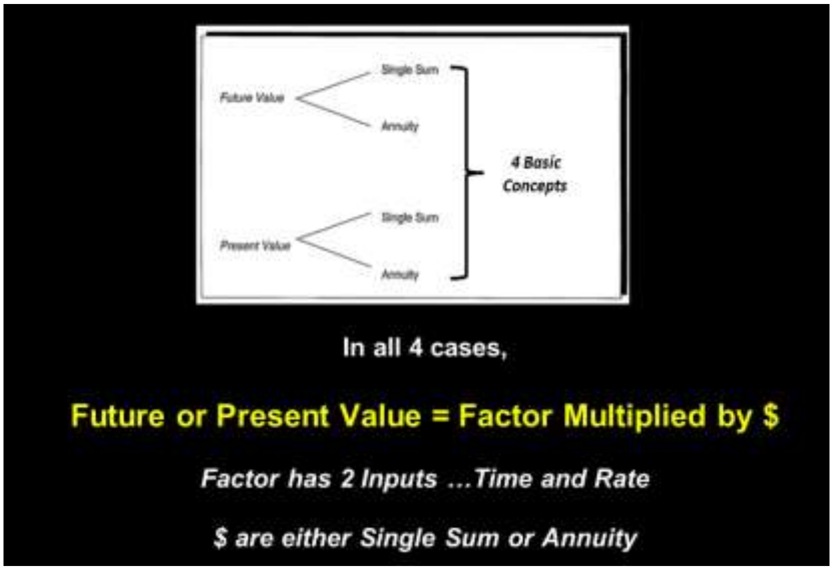
Four Basic Concepts of Time Value of Money



For each of the four concepts in Exhibit 1, there are two critical additional inputs needed to solve any time value of money problem. The first is the rate of return, and the second is the time dimension. A precise rate of return and time dimension will give us what is referred to as a factor for a specific type of time value of money problem. To find our solution, we multiply the dollars involved (either single sum or annuity) by that factor. This is shown in Exhibit 2.

Exhibit 2

Solving a Time Value of Money Problem



In all 4 cases,

Future or Present Value = Factor Multiplied by $

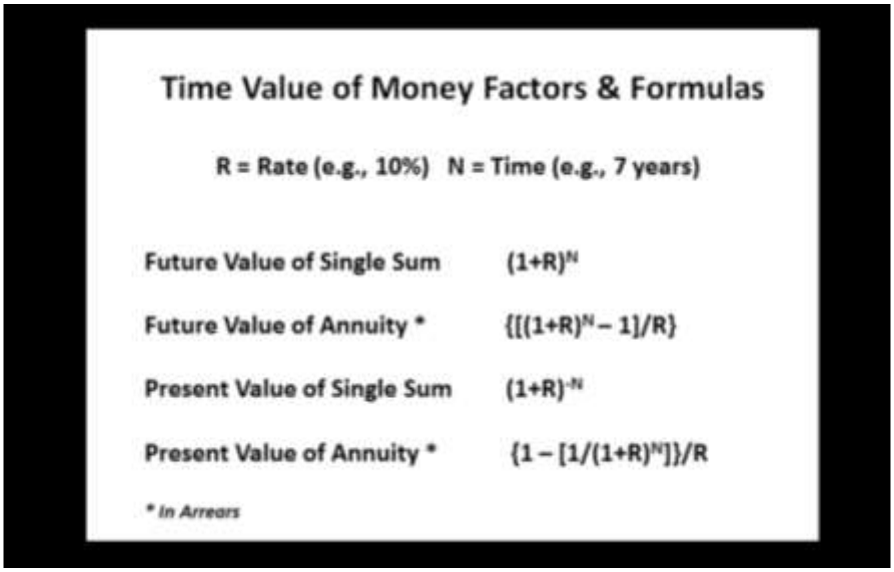
Factor has 2 Inputs ...Time and Rate

$ are either Single Sum or Annuity

Behind all of the factors are the TRI Critical Equations that are among the most well known in finance. Exhibit 3 provides the equations to solve for any factor. These are the formulas that are built into the solutions found in time value of money tables and in the coding of financial calculators and/or spreadsheets like Excel. R is the rate of return and N is the time horizon. While we state R as a percentage (e.g., 10% in the Exhibit) in the formula it would be input as a decimal (e.g., .10).

Exhibit 3

TRI Critical Equations for Time Value of Money



Time Value of Money Factors & Formulas

R = Rate (e.g., 10%)   N = Time (e.g., 7 years)

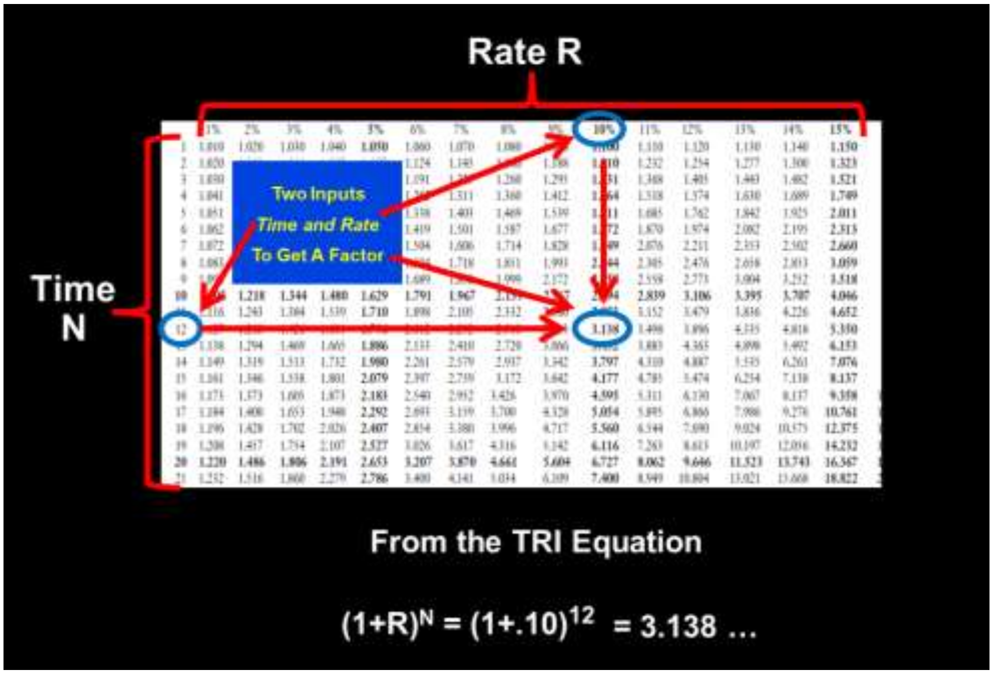
Future Value of Single Sum	$(1+R)^N$
Future Value of Annuity *	$\{[(1+R)^N - 1]/R\}$
Present Value of Single Sum	$(1+R)^{-N}$
Present Value of Annuity *	$\{1 - [1/(1+R)^N]\}/R$

** In Arrears*

Numerical values for the factors of all four cases in Exhibits 1, 2 and 3 can be found in time value of money tables. Typically, you will find the rate across the top and the time down the left-hand side. By identifying the rate and time, you can go to the intersection and determine the factor. Because rate and time are usually rounded to whole numbers in the tables, they can provide a reasonable approximation of the factor. The equations, on the other hand, are valuable for arriving at precise answers for problems that involve non-whole numbers, like a rate of 8.76% and time horizon of 22.5 years. Exhibit 4 demonstrates the use of one of the tables. This example is specfically for future value of a single sum but the other three work in an identical fashion. The four time value of money tables that correspond to Exhibits 1, 2 and 3 can be found in the appendix.

Exhibit 4

Time Value of Money Table for Factor Determination



Rate R

Time N

Two Inputs
Time and Rate
To Get A Factor

From the TRI Equation

$(1+R)^N = (1+.10)^{12} = 3.138 \ldots$

As the analysis has been presented to date, there are four numbers in any problem (rate, time, $ as single sum or annuity, and the solution). Once you know three of the four constants, the fourth can always be calculated. As an example, we might want to know how long it would take $100 to triple to $300 if our return was 12%, or what rate we would need to earn to double our money over 10 years. Any combination is feasible to ask questions around.

The next four sections are devoted to numerical applications of the equation.

Sample Problems – Future Value of a Single Sum - $(1+R)^N$

The following examples are solved via the tables. Try the formulas as practice.

How much will $100 invested today at 7% per year be worth by the end of four years?

$$\text{Future Value of a Single Sum} = (1 + .07)^4 = 1.311$$

$$\text{Future Value} = \$100 \text{ X } 1.311 = \$131.10$$

Note, in future value of single sum problems, there is an implicit assumption that no monies are removed over time. This is the concept of compounding interest, or interest on interest. In our first problem, this means the total $107 at the end of year one is reinvested at 7% for the second year. In the second year, we earn not only the 7% on the original $100 but 7% on the $7 (or 49 cents) of interest earned in the first year. This is why the 7% two-year factor is 1.145. The table values are truncated at three decimals. Graphically this would be a non-linear relationship to indicate the constant growth.

How much will $1,000 invested today at 10% per year be worth after 20 years?

$$\text{Future Value of a Single Sum} = (1 + .10)^{20} = 6.727$$

$$\text{Future Value} = \$1{,}000 \text{ X } 6.727 = \$6{,}727$$

Note that in the example the compounded interest over 20 years dwarfs the initial investment of $1,000. This is an example of the power of time with continual reinvestment.

How much would $100 be worth if it is invested today for five years at 10%, and the full amount at end of the fifth year reinvested for the next 10 years at 5%? That is, how much would you have at the end of 15 years? Test your knowledge and prove the answer is approximately $262. After you are satisfied with the $262, ask yourself how much would $100 invested today for 10 years at 5% and the full proceeds at end of year 10 reinvested for the next five years at 10% be worth at the end of 15 years? That is, as in the prior problem, how much would you have at the end of 15 years?

Before you jump to the tables, a calculator or the formulas, think logically what the answer has to be. The answer is still $262 because these are just multiplicative relationships. Does this mean you would be indifferent to an offer of both options? If your only concern was terminal value (how much you have at the end of 15 years), you would be indifferent. What if you had the option of getting out of either scenario before year 15? Would you still be indifferent? The answer is no. In this problem, the first scenario will always have a larger value up to that last second before the end of year 15. Prove it.

One of the most common heuristics involving the time value of money, and the future value of a single sum, is the ***"Rule of 72."*** The essence of this rule is that by dividing 72 by the rate, you will get a reasonable approximation of how long it takes to double your money. For example, with a 10% annual rate, divide 72 by 10, and the answer is 7.2 years. A more precise answer, arrived at via our formulas, would be 7.273 years.

Sample Problems – Future Value of an Annuity - $\{[(1+R)^N – 1]/R\}$

How much would you have if you invested $1,000 per year for four years (with the first annuity payment at the end of year one or in arrears) if the rate was 10%?

$$\text{Future Value of an Annuity} = \{[(1 + .10)^{4} -1] / .10\} = 4.641$$

$$\text{Future Value} = \$1,000 \text{ X } 4.641 = \$4,641$$

This problem could also be solved as four single sums of $1,000 each for their respective time. The fourth payment of $1,000, under the assumption of arrearage, would not earn any interest because the process immediately stops at the end of the fourth year, before any interest can be earned on it.

How much would you have if you invested $2,000 per year for 30 years (with the first annuity payment at the end of year one or in arrears) if the rate was 12%?

$$\text{Future Value of an Annuity} = \{[(1 + .12)^{30} - 1] / .12\} = 241.333$$

$$\text{Future Value} = \$2,000 \text{ X } 241.333 = \$482,665$$

In this example, you can vividly see the impact of time and rate on final value. Over the 30 years, $60,000 was invested. The difference of $422,665 is all of the accumulated return. This is a very common retirement scenario. We chose 12% because many academics would suggest this is a proxy for long-term stock market returns. The 30 years corresponds to a working career and the buildup phase of retirement monies.

Sample Problems – Present Value of a Single Sum - $(1+R)^{-N}$

What is the present value of $1,000 to be received in three years at 10%?

$$\text{Present Value of a Single Sum} = (1 + .10)^{-3} = .751$$

$$\text{Present Value} = \$1{,}000 \times .751 = \$751$$

In this example, you have the right to receive $1,000 three years from today. The question being answered is, what would you pay for this right? In a perfect world, if 10% represented your true required return, you would be indifferent to the choice of an offer of $751 today or $1,000 in three years. Consider turning this into a future problem of a single sum. If you invested $751 today at 10% you would have $1,000 in three years. Of course, if 10% was not your opportunity cost of money, then your decision may change.

The most important takeaway of this problem is to understand that **present values are market values.** The question “What is the present value?” is equivalent to asking “What is the market value?” or “How much would you pay today?” This is the essence of why present value concepts are used in all types of valuation when there is a time dimension to when monies are received.

Our next example is designed to demonstrate how to deal with the present value of a series of divergent monies over time. This is a very common real-world business application. Your business is doing an appropriation request and the finance team has generated the following cash flow stream:

End of Year	Cash Flow
1	$100
2	$200
3	$0
4	$100

Your required rate of return (an example would be the cost of capital in TRI Equation 4) is 8%. What is the present value of the cash flow? The key is to recognize that each cash flow can be treated as an individual single sum. Calculate the factors using the present value of a single sum formula, $(1+R)^{-N}$, and create a chart as follows:

End of Year	Cash Flow	X	Factor	=	Present Value
1	\$100	X	.926	=	\$92.60
2	\$200	X	.857	=	\$171.40
3	\$0	X	.794	=	\$0
4	\$100	X	.735	=	\$73.50
					\$337.50 ======

The present value of the sum is the sum of the individual present values, and it equals \$337.50. (This approach is also readily and easily applied in a spreadsheet format.) Therefore, \$337.50 is the maximum amount the company could invest and still earn an 8% return. If we pay less than \$337.50 and generate the expected cash flows, our return exceeds the 8%. If we pay more and generate the cash flows as given our return will be less than the 8%. This is the logic behind all NPV and IRR problems.

Sample Problems – Present Value of an Annuity - $\{1 - [1/(1+R)^N]\}/R$

What is the present value of receiving \$1,000 per year (payment in arrears) for five years at 12%?

$$\text{Present Value of an Annuity} = \{1 - [1 / (1 + .12)^5]\} / .12 = 3.605$$

$$\text{Present Value} = \$1{,}000 \text{ X } 3.605 = \$3{,}605$$

This problem could also be solved as five individual present values of single sums of \$1,000 each for their respective time.

If the rate in our problem was 15%, holding everything else constant, what is the present value?

$$\text{Present Value of an Annuity} = \{1 - [1 / (1 + .15)^5]\} / .15 = 3.352$$

$$\text{Present Value} = \$1{,}000 \text{ X } 3.352 = \$3{,}352$$

Note, when the rate goes up the present value goes down. This is a fundamental result with positive rates:

Rates Up … Present Value Down
Rates Down… Present Value Up

There is a very interesting problem in present value space when the annuity is to be received forever. A perpetual annuity is often referred to as perpetuity. The implication in our formulation is that N goes to ∞.

$$\text{If } N \text{ goes to } \infty \text{ then } \{1 - [1/(1+R)^N]\}/R \Rightarrow 1/R.$$

The implication for valuing perpetuity is that we divide the perpetual cash flow by the rate. As an example, what is the present value of receiving $100 per year forever, if you require a 10% return?

$$\text{Present Value} = \$100/.10 = \$1,000$$

If you converted this problem into a future value problem, imagine making an investment of $1,000 earning exactly 10% each year forever. There is an assumption of yearly compounding. At the end of year one you could withdraw the $100 of interest earned and have the $1,000 investment for year two. At the end of year two you could withdraw another $100, and so on and so forth. You could forever make withdrawals of the $100 per year (at the end of the year) and never dip into your original investment. Perpetuity valuation can occur with consols (perpetual bonds issued in Europe to fight wars in antiquity) and preferred stock as examples. In appropriation requests, it is also common after a period of growth to assume a terminal value based upon a perpetuity formulation. This assumption is made to reflect simplicity and maybe a conservative case, and also because the forecasts beyond the growth period (typically five to 10 years in practice) can be fraught with significant error.

Annual to Continuous Compounding

All of the equations and examples thus far have assumed an annual rate paid at the end of the year on the beginning balance. There are other rates that assume other-than-annual compounding. This is often the case in financial service markets and deposits. For example, you may have heard the statement “7% annual rate with monthly compounding.” The number of times compounding will occur within a year is given by M. Any of the formulas in Exhibit 3 can be adjusted to allow for M-period compounding by dividing the R by M and multiplying the N by M.

The formula for the future value of a single sum is given by:

$$(1+R)^{N}.$$

The formula for the future value of a single sum with M period compounding is therefore given by:

$$(1+R/M)^{NM}.$$

With an annual rate of 10% and five years, the factor is 1.611 (from the formula or table in the appendix). With monthly compounding (i.e., M = 12) the factor is 1.645. A $100,000 investment under annual compounding would grow to $161,100. With monthly compounding the terminal value would be $164,500, or $3,400 more, as a result of the frequency of compounding.

The limit condition of compounding would occur when M goes to infinity. In financial theory this is defined as continuous compounding. The formula for the future value of a single sum with continuous compounding is given by:

$$e^{RN}$$

where e is Napier's number, or the base of the natural logarithm. An approximate value of e is 2.713. As esoteric as the equation may seem, continuous compounding has many applications in finance. The last time we saw the concept was in TRI Equation 6 on Options. It is embedded within the Black-Scholes formula.

One needs to be very careful when comparing rates with divergent compounding periods; for example, 6% compounded semi-annually vs 5.8% compounded daily. We can only make the comparison after calculating what is often called the "effective annual rate" or "effective (or equivalent) annual rate." The mathematics is detailed below using the M period compounding model above. Six percent compounded semi-annually equates to an effective annual rate of 6.09%, while 5.8% compounded daily (using 365 days) equates to an effective annual rate of 5.97%. Only the effective annual rates can be truly compared.

$$\text{Effective Annual Rate} = [1 + (R / M)]^{M} - 1$$

$$\text{EAR} = [1 + (.06 / 2)]^{2} - 1 = .0609 = 6.09\%$$

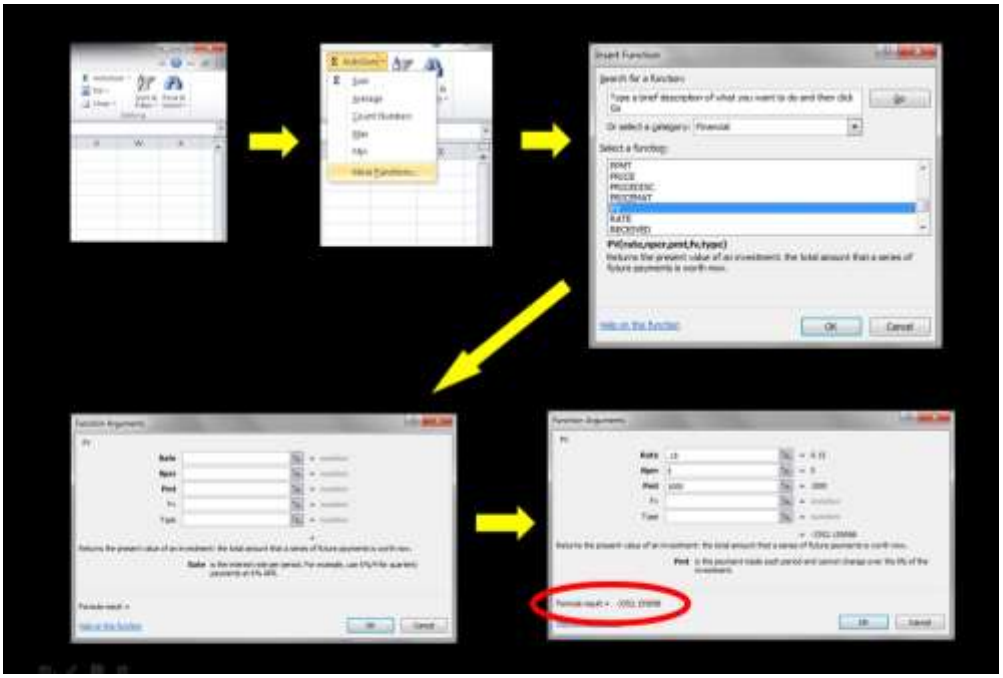
$$\text{EAR} = [1 + (.058 / 365)]^{365} - 1 = .0597 = 5.97\%$$

Excel Applications

In this section, we review the basics of solving a present value problem in Excel. The example in Exhibit 5 is specifically for present value. It outlines the steps to solve the present value of the problem above, where we have the right to receive $1,000 per year for five years at 15%. Remember that we calculated that present value at $3,352. Note the Excel answer enclosed in the red ellipse is -$3,352. The interpretation of the minus sign is that this is the amount we would pay today to have the right to receive $1,000 per year for five years if we required a 15% return.

Exhibit 5

Excel Applications of Present Value



We leave it to the reader to practice with the Excel time value of money functions. Excel is extremely robust with regard to all financial functions.

Valuing a Bond

The valuation of a corporate bond is an excellent example of the mathematics of present value applied to a financial instrument. An investor in a bond typically invests $1,000 per bond at original issuance of the bond by the corporation. The investor receives an interest amount per year (in practice semi-annual), which is referred to as the coupon rate. The bond will have a maturity and, if all goes well, will be repaid in full at the original investment at that time.

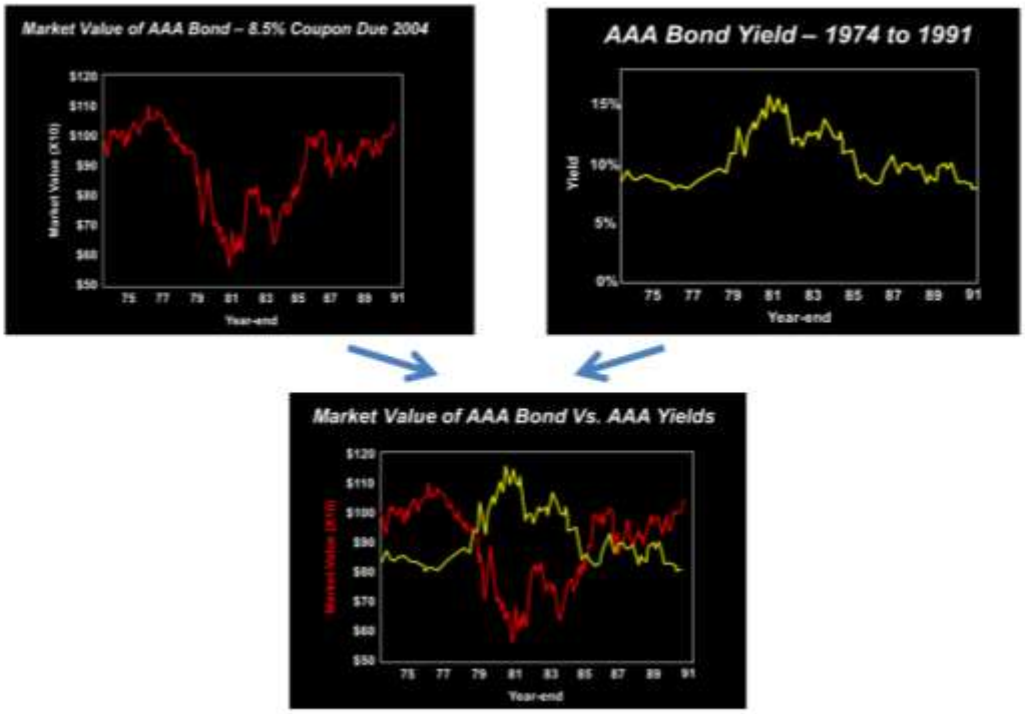
Exhibit 6 is the market value of a bond for a AAA-rated company. The bond was originally issued in 1975 and matures in 2004; that is, it has a 30-year maturity. When the bond was originally issued, 30-year AAA interest rates were 8.5%. In the ideal world, the investor would give the company $1,000 and expect to receive $85 per year and $1,000 at the end of the 30 years. Corporate bonds typically trade, and rarely will investors hold them until maturity. Any trades will be at a fair market value. Even if the creditworthiness of the business does not change (i.e., remains AAA rated), if interest rates change after the original issuance, the market value may deviate from the $1,000 face value per bond.

The primary reason AAA rates can vary over time is that U.S. government debt (with a remaining identical maturity) can vary due to changes in macro-economic conditions and inflationary expectations. That is, the government yield curve will have shifted. The market value of the bond at any time, prior to maturity, is the present value of the remaining coupon payments (in this case the $85 per year) and the present value of $1,000 to be received at maturity. Technically, if the bond only has 20 years left, the market value must be reflective of the 20 years, not the original 30.

The upper left-hand corner of Exhibit 6 shows the market value of the aforementioned bond, the upper right-hand corner shows AAA bond yields, and the lower middle shows an overlay of both graphics. The key is the inverse relationship between rates and value, which we articulated earlier. The bond markets of the world are an area where the concept manifests itself in trading all the time.

Exhibit 6

The Inverse Relationship between Rates and Value in the Bond Market



The market value of the bond dropped to under $600 by the early 1980s even though the credit quality of the issuer of the bond remained AAA. This was a period of significant (and for the U.S. unprecedented) inflation. The inflation expectations exceeded 12% at the time, and interest rates increased. You will note the example ends in the early 1990s. Interest rates on AAA debt had dropped to below 8.5% and it was an opportunity for the company to call the debt (essentially refinance it at lower rates). As we discussed in TRI Equation #6 on Options, the opportunity to call requires the payment of a premium. In effect, this is what homeowners do all the time, typically without any pre-payment penalty, but at a much larger corporate level.

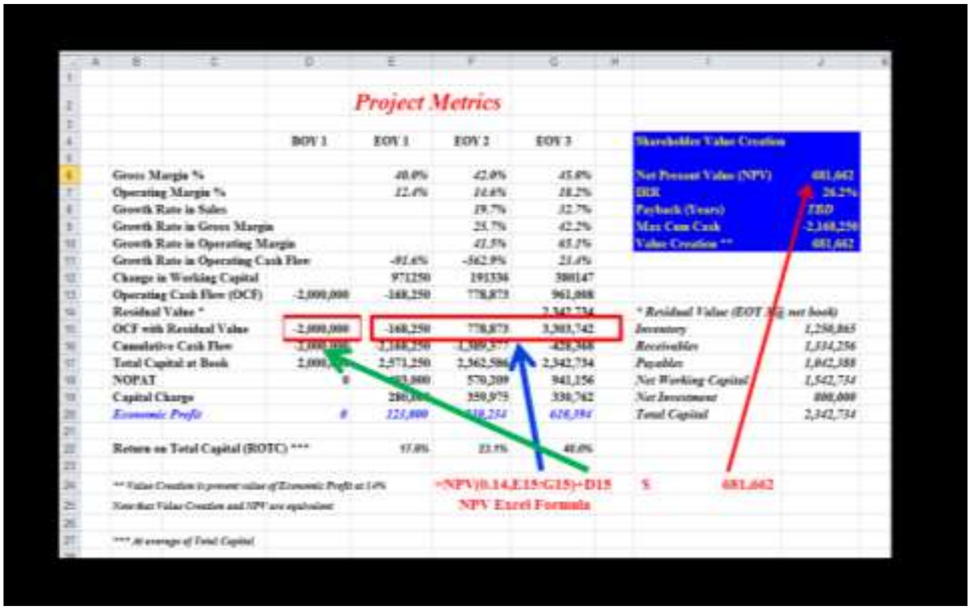
Time value of money is basic to valuing bonds. At the time of this writing, many (but not all) investment experts believe that an increase in interest rates is inevitable coming off of global monetary easing and that bond markets could drop significantly in value. We will check back on this one in a few years. Negative interests are also a current reality in capital markets and, while beyond our scope of detail, do create many unique mathematical relationships in the time value of money.

Valuing a Capital Investment (or Application to an Appropriation Request)

In TRI Equation #9 on Cash Flow, we presented a sample appropriation request of Hypo-Product. This was a standard free cash flow application. Exhibit 20 in TRI Equation #9 showed a summary of many relevant metrics, including net present value, and is reproduced as Exhibit 7 below, with Excel modifications to demonstrate the present value syntax. The cost of capital was 14%. Note the NPV formula has two pieces (the present value of cash flows and the initial investment assumed paid today).

Exhibit 7

NPV using Excel



Project Metrics

	BOY 1	EOY 1	EOY 2	EOY 3
Gross Margin %		40.0%	42.0%	45.0%
Operating Margin %		12.4%	14.6%	18.2%
Growth Rate in Sales			19.7%	32.7%
Growth Rate in Gross Margin			25.7%	42.2%
Growth Rate in Operating Margin			41.5%	65.1%
Growth Rate in Operating Cash Flow		-91.6%	-562.9%	23.4%
Change in Working Capital		971250	191336	380147
Operating Cash Flow (OCF)	-2,000,000	-168,250	778,873	961,008
Residual Value *				2,342,734
OCF with Residual Value	-2,000,000	-168,250	778,873	3,303,742
Cumulative Cash Flow	-2,000,000	-2,168,250	-1,389,377	-428,368
Total Capital at Book	2,000,000	2,571,250	2,362,586	2,342,734
NOPAT	0	403,000	570,209	941,156
Capital Charge		280,000	359,975	330,762
Economic Profit	*0*	*123,000*	*210,234*	*610,394*
Return on Total Capital (ROTC) ***		17.6%	23.1%	40.0%

Shareholder Value Creation	
Net Present Value (NPV)	681,662
IRR	26.2%
Payback (Years)	TBD
Max Cum Cash	-2,168,250
Value Creation **	681,662

* Residual Value (EOT 3 @ net book)	
Inventory	*1,250,865*
Receivables	*1,334,256*
Payables	*1,042,388*
Net Working Capital	*1,542,734*
Net Investment	*800,000*
Total Capital	*2,342,734*

*** Value Creation is present value of Economic Profit at 14%*

Note that Value Creation and NPV are equivalent

=NPV(0.14,E15:G15)+D15 $ 681,662

NPV Excel Formula

**** At average of Total Capital*

Valuing a Stock

Equity markets are intrinsically forward looking, and in the spirit of efficiency, prices should reflect all relevant information. The forward-looking nature of equity markets makes them potentially ideal to use time value of money techniques, specifically present value, to determine the market value.

Financial and investment analysts will routinely use present value techniques from the very simple to the incredibly complex. One of the basic models for valuing equity is referred to as a constant growth model. In this model, the most common application is that an estimate of the price per share is the present value of future dividends. (Naturally, there are variations for non-dividend-paying companies, as well as discounted cash flow models). In the constant growth model, the dividend is assumed to grow at a constant rate per year forever. (There are also multistage dividend growth models that allow for growth rates to regress to regular growth in the economy.) The rate used is the cost of equity capital as discussed in TRI Equation #4, with the CAPM being the most widely used in practice. The dividend growth model in simplest form is given by:

$$P = D/(K_e - g)$$

Where P is the price per share, D is expected dividend per share at the end of year one, K_e is the cost of equity, and g is the growth of dividends per year forever. If D was \$1.00, $K_e = 10\%$ and $g = 2\%$, the estimate of the stock price would be:

$$P = \$1.00/(.10-.02) = \$1.00/.08 = \$12.50$$

In the price per share formulation above, note that when g = 0% (that is, no growth), the formula simplifies to D/K_e. This is our aforementioned perpetuity formulation derived from the present value of an annuity. In this case, the annuity is the D per year forever.

Valuing the Decision to Take a Discount

One of the timeless problems in finance is determining the optimal timing to take a discount or not with trade credit. For example, if you were offered 2/10 N/30 terms, (taking a 2% discount to pay within 10 days, or paying the total in 30) and your cost of borrowing was 10% on an annual basis, does it pay to take the discount even if you have to borrow? The basic answer is nearly always yes, because the opportunity cost of not taking the discount is often calculated at approximately 36%. By taking advantage of a discount, a customer is essentially changing the total amount due into two components: interest paid on the invoice and the revised balance due to the vendor. For example, a \$5,000 net due amount would now be considered \$100 of interest, and \$4,900 of amount due for the goods or services. This basic solution is given by 2/98 times 360/20.

$$(2/98) \times (360/20) = 36.7\%$$

The reality is that when the time value of money is considered, the cost of not taking the discount is given by using the logic of $(1+R)^N$.

$$[1+(2/98)]^{(360/20)} - 1 = 43.8\% \text{ as a \%.}$$

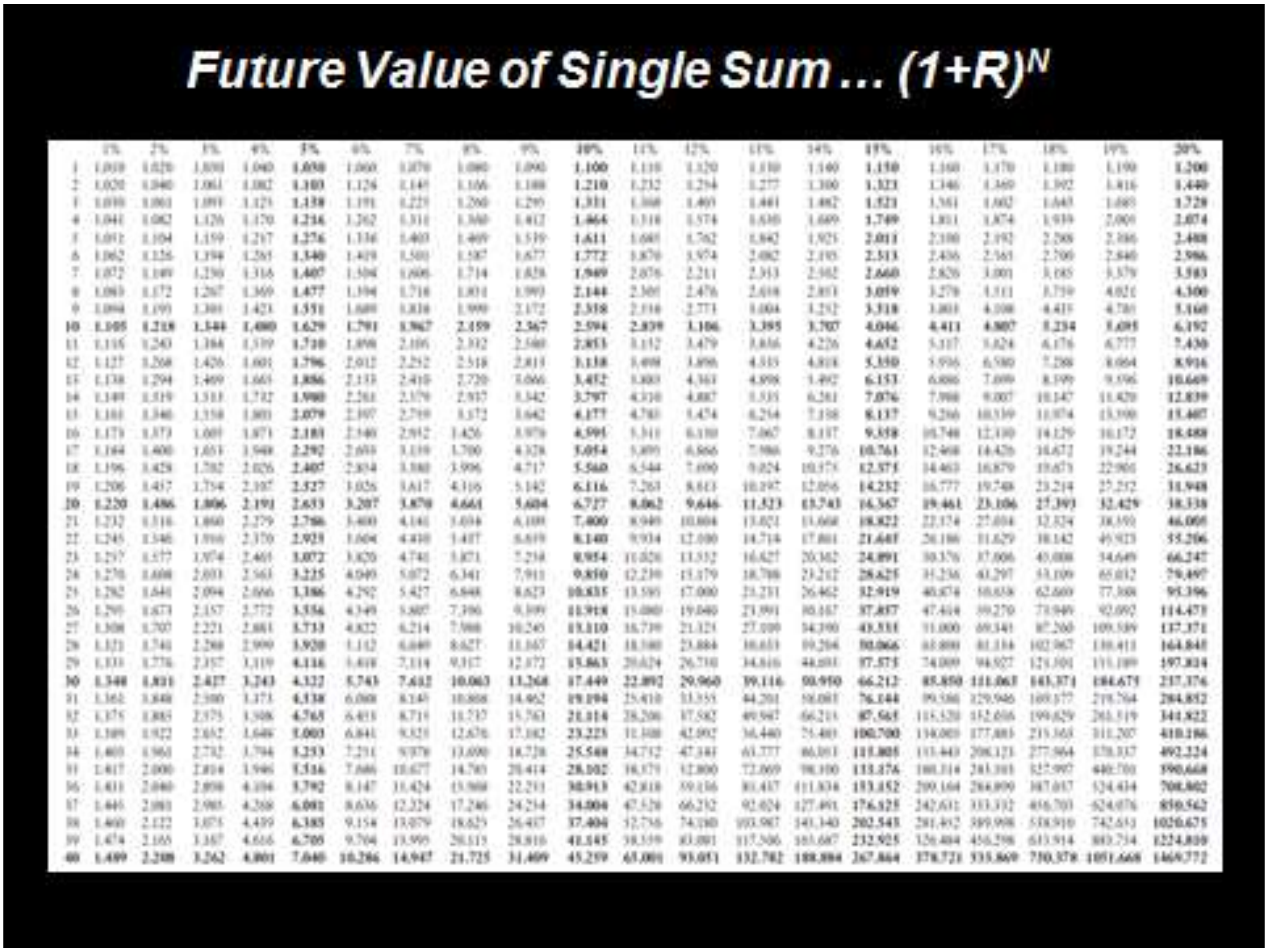
The true cost of not taking the discount is significantly higher than the routine 36% tossed around in examples. A key lesson is to make sure you are taking the discounts when the math is in your favor.

Summary

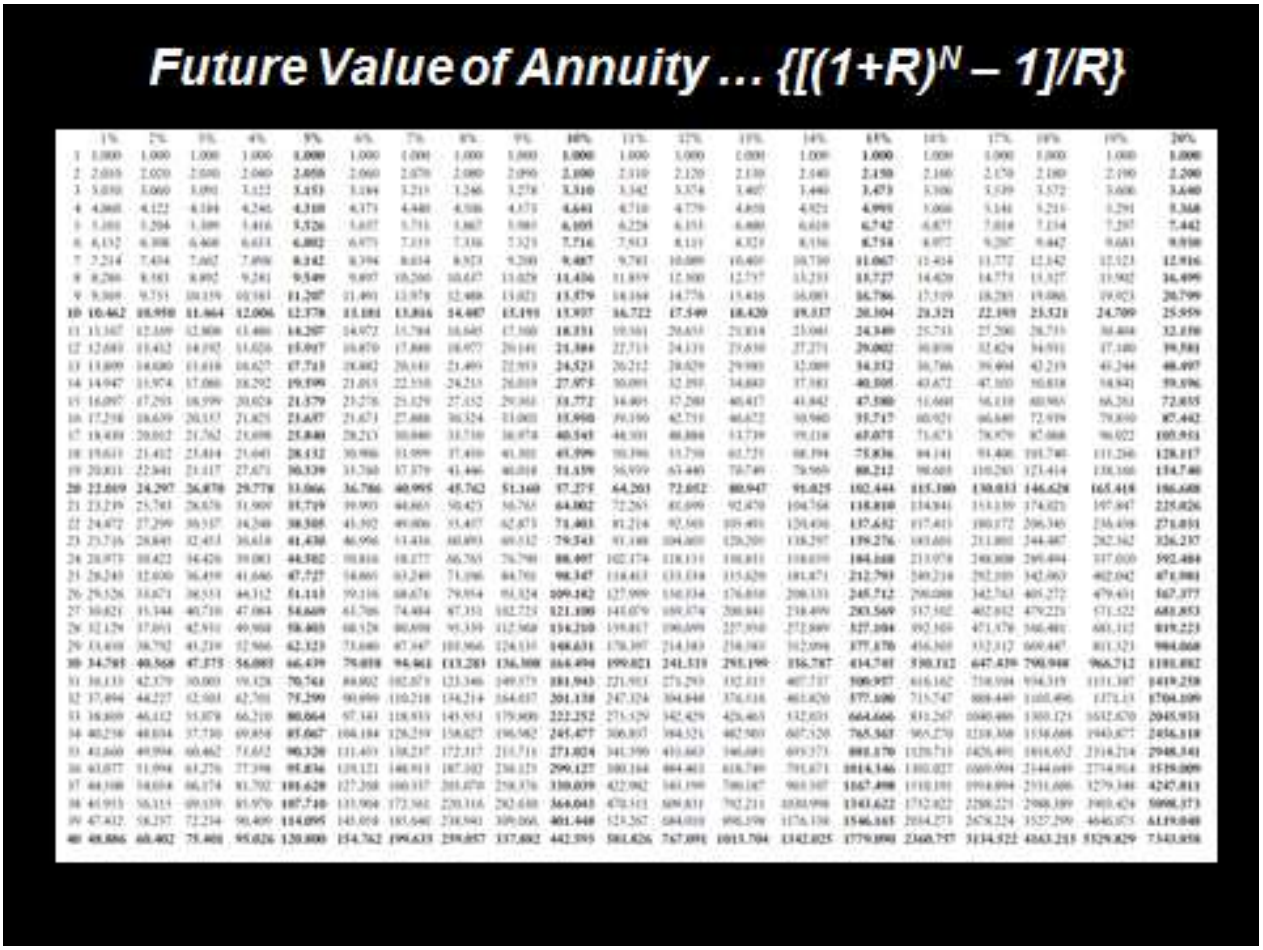
While Einstein may have overstated when he said ***"the most powerful force in the universe is compound interest,"*** the time value of money *is* fundamental to all aspects of business decision-making. We have tried to provide insight into the basics of this concept through TRI's Critical Equation #10.

Appendix – Time Value of Money Tables

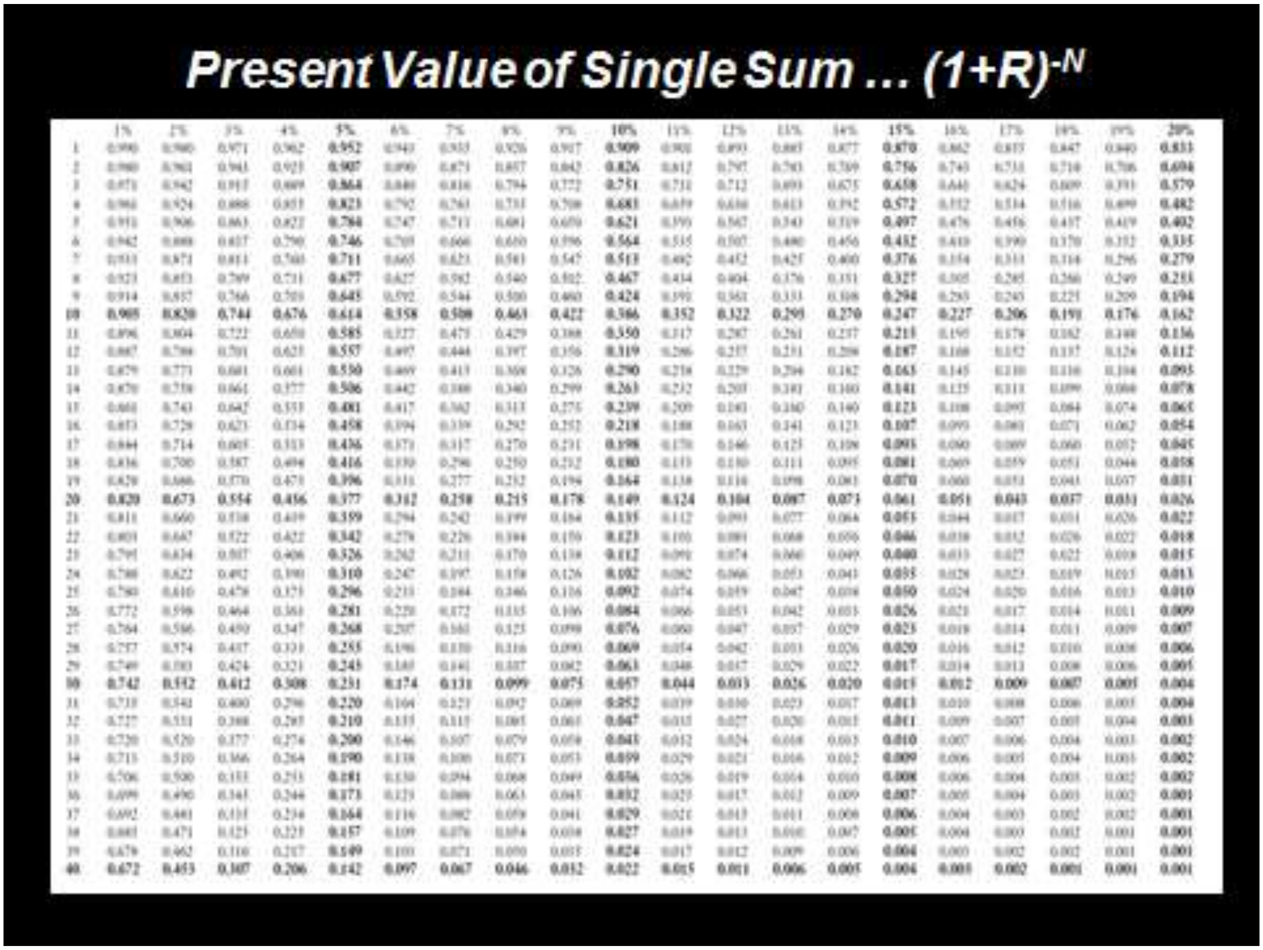
Presented below are the four time value of money tables as discussed in Exhibits 1, 2, 3 and 4.



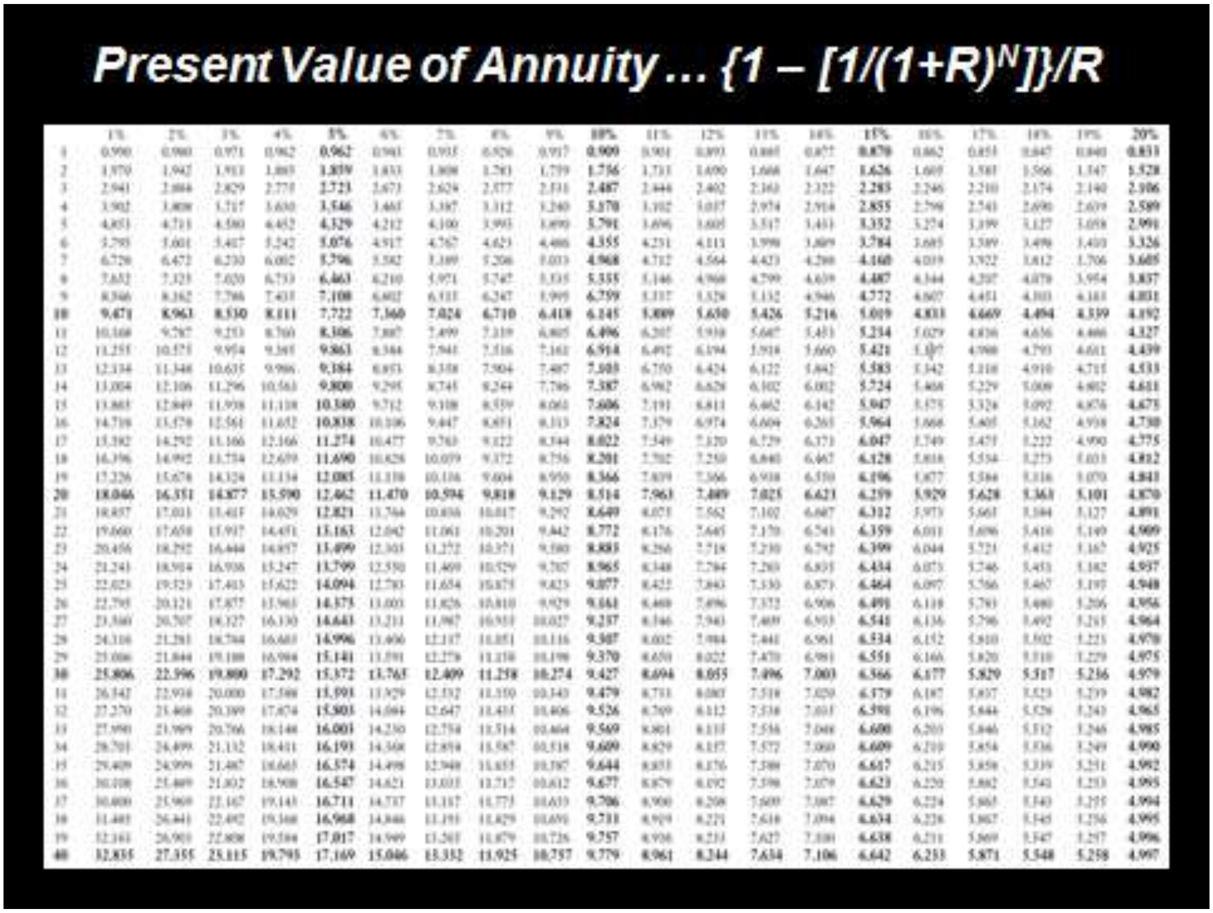
Future Value of Single Sum … $(1+R)^N$



Future Value of Annuity … $\{[(1+R)^N - 1]/R\}$



Present Value of Single Sum … $(1+R)^{-N}$



Present Value of Annuity… {1 – [1/(1+R)^N]}/R